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Calculation by series expansion of the heat transfer in laminar, constant-property boundary layers at nonisothermal surfaces

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With 2 figures in the text

SUMMARY

A constant-property analysis of the heat transfer at a nonisothermal surface of a blunt body has been made by means of series expansions of Blasius type for two dimensional and axially symmetrical laminar flow. The temperature of the surface and the velocity at the edge of the boundary layer are assumed to have power series distributions. Some new terms, containing universal functions, must be added to the existing solutions for heat transfer at isothermal surfaces, earlier published by the author. Some of these functions have been calculated numerically.

1. Introduction

The problem of calculating the heat transferred by forced convection from a body with a temperature varying over its surface, can arise under several circumstances. Thus, even at stationary conditions, a nonuniform heating or cooling from inside or outside of the body can result in a nonuniform temperature distribution owing to the limited heat conductivity of the matter. However, whether the real temperature of the surface becomes practically uniform or not, the problem mentioned will arise if the velocity of the flow is so large that heating effects due to internal friction and to pressure changes are noticeable. At not too large velocities the energy equation describing the temperature field remains linear, the inhomogeneous terms giving account of those heating effects. A general solution can then be obtained by adding a particular solution of the complete equation and a general solution of the homogeneous part. This homogeneous part is the equation of low speed heat transfer [1]. For the former solution, that of the adiabatic wall (that is, that of a nonheated, insulated body) is useful. The total heat transfer will then be described by the homogeneous equation and will depend on the difference between the actual wall temperature and that of the adiabatic case. At moderate velocities the properties of the fluid can be treated as constants [2].

Much work has been done on exact solutions of the heat transfer at nonisothermal surfaces of plates, wedges and cones (e.g. [3], [4], [5]). For calculating boundary layers at bodies of arbitrary shape, there are available several approximate methods which, however, require a further assumption together with the fundamental equations. For getting solutions without such assumptions, a series expansion with respect to the distance along the wall can be made. It can be continued by some step-by-step method. For the thermal boundary layer at vanishing viscous heating and isothermal surfaces, such a series solution has been worked out for blunt bodies by the author [6] for two-dimensional and axially symmetrical flows. In the former case the solutions given by Blasius [7], Hiemenz [8], and Howarth [9] were used for the velocity field. The corresponding solutions for the viscous heating at adiabatic walls have been studied by the author [10]. The series solution given by [6] can be extended to a constant-property solution in the nonisothermal case, still using universal functions independent of the particular data of the problem. A brief account of this is given below. The transition from the dimensioned quantities to the dimensionless universal functions has now been made in a way somewhat different from that of paper [6]. Since the equations of heat and mass transfer are quite the same in this case, the analysis can be used also for mass transfer. Stationary conditions are assumed.

2. Series expansions

A. *Two-dimensional symmetrical case*

The velocity- and temperature fields are described by the equations of momentum, continuity, and energy; and by the corresponding boundary and initial conditions. For a laminar boundary layer the energy equation for a constant-property analysis at neglected viscous heating is

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=a\frac{\partial^2 t}{\partial y^2}; \tag{1}$$

(x, y and u, v = coordinates and velocity components parallel and perpendicular to the wall; a = thermal diffusivity $=\lambda/\varrho c_p$; t = temperature excess with respect to the surroundings). The velocity U outside the boundary layer is assumed to be

$$U=u_1 x+u_3 x^3+u_5 x^5+\cdots \tag{2}$$

(the leading term indicating a blunt body). The boundary conditions are:

$$\text{if } y=0,\ t \text{ prescribed; and if } y=\infty,\ t=0. \tag{3}$$

For the isothermal case (prescribed value = constant $=t_0$) ref. [6] gives the solution

$$\frac{t}{t_0}=c_0+c_2 x^2+c_4 x^4+\cdots \tag{4}$$

$$\frac{t}{t_0}=F_0+\frac{4\,u_3 F_2}{u_1}x^2+\frac{6\,u_5}{u_1}\left(G_4+\frac{u_3^2}{u_1 u_5}H_4\right)x^4+\frac{8\,u_7}{u_1}\left(G_6+\frac{u_3 u_5}{u_1 u_7}H_6+\frac{u_3^3}{u_1^2 u_7}K_6\right)x^6+\cdots. \tag{5}$$

The quantities with capital letters are universal functions depending (besides on the Prandtl number $\sigma = \nu/a$) only on $\eta = y\sqrt{u_1/\nu}$; (ν = kinematic viscosity). For those functions the following equations are valid.

$$\left.\begin{aligned}
&\frac{1}{\sigma}F_0'' = -f_1 F_0' \\
&\frac{1}{\sigma}F_2'' = -f_1 F_2' + 2 f_1' F_2 - 3 f_3 F_0' \\
&\left.\begin{aligned}
&\frac{1}{\sigma}G_4'' = -f_1 G_4' + 4 f_1' G_4 - 5 g_5 F_0' \\
&\frac{1}{\sigma}H_4'' = -f_1 H_4' + 4 f_1' H_4 - 5 h_5 F_0' + \frac{8}{3}(2 f_3' F_2 - 3 f_3 F_2')
\end{aligned}\right\} \\
&\left.\begin{aligned}
&\frac{1}{\sigma}G_6'' = -f_1 G_6' + 6 f_1' G_6 - 7 g_7 F_0' \\
&\frac{1}{\sigma}H_6'' = -f_1 H_6' + 6 f_1' H_6 - 7 h_7 F_0' + 3(4 f_3' G_4 + 2 g_5' F_2 - 3 f_3 G_4' - 5 g_5 F_2') \\
&\frac{1}{\sigma}K_6'' = -f_1 K_6' + 6 f_1' K_6 - 7 k_7 F_0' + 3(4 f_3' H_4 + 2 h_5' F_2 - 3 f_3 H_4' - 5 h_5 F_2')
\end{aligned}\right\}
\end{aligned}\right\} \tag{6}$$

The functions $f_1 \ldots$ (small letters) are the velocity functions of Howarth [9]. The boundary conditions of the equations (6) are $\eta = 0$, $F_0 = 1$ and other functions $= 0$; $\eta = \infty$, all functions $= 0$. Some of the functions are calculated numerically and given in [6].

The heat transfer is described by the Nusselt number Nu. If D and t_m are the reference quantities for getting a dimensionless number, we have

$$Nu = \frac{D}{t_m}\left(-\frac{\partial t}{\partial y}\right)_{y=0}; \tag{7}$$

Nu can be calculated from equations (5), (6), (7).

The solution described can be generalised to the nonisothermal case, if the prescribed temperature distribution over the surface is of power series type:

$$t = t_0 + t_2 x^2 + t_4 x^4 + \cdots . \tag{8}$$

Of course the equations (6) could be solved directly by satisfying the particular coefficient values of (8), but this would not give any universal functions independent of the particular data of the problem. Instead of using this way we benefit by the fact that the energy equation now is linear and therefore solutions can be superimposed. We therefore study for $n > 0$ the distribution

$$t = t_n x^n \quad (n \text{ an even number}). \tag{9}$$

Since the equations of the velocity field are not linear we cannot superimpose solutions made for the single terms of (2), that is for wedge solutions. Now for studying the distribution (9) we can still use (5) and (6), together satisfying (1), only changing in (5) the constant t_0 for

$$t_n \frac{u_1}{u_{n+1}} \tag{10}$$

and giving some of the functions new boundary values at the wall. Thus for $\eta = 0$, the first function in the factor for x^n gets the value $1/(n+2)$ and all other functions vanish. The former function is called C_n. Some of the latter ones are defined by homogeneous equations and will therefore, according to the boundary values, become identically equal to 0. Those that do not disappear but possibly get their equation slightly changed, are marked by a circle (for example $\mathring{K}_6$).

By summing solutions for different n-values we get the solution for the distribution (8):

$$\begin{aligned} t = t_0 \left[F_0 + \frac{4 u_3}{u_1} F_2 x^2 + \cdots \right] + t_2 \Bigg[4 C_2 x^2 + 6 \frac{u_3}{u_1} \mathring{H}_4 x^4 + \\ + 8 \left(\frac{u_5}{u_1} \mathring{H}_{6a} + \frac{u_3^2}{u_1^2} \mathring{K}_6 \right) x^6 + \cdots \Bigg] + t_4 \left[6 C_4 x^4 + 8 \frac{u_3}{u_1} \mathring{H}_{6b} x^6 + \cdots \right] + \\ + t_6 \left[8 C_6 x^6 + \cdots \right] + \cdots . \end{aligned} \tag{11}$$

Of course the terms in (11) could be collected according to powers of x. The functions are defined by the equations

$$\left.\begin{aligned} \frac{1}{\sigma} C_2'' &= - f_1 C_2' + 2 f_1' C_2 \\ \frac{1}{\sigma} C_4'' &= - f_1 C_4' + 4 f_1' C_4 \\ \frac{1}{\sigma} \mathring{H}_4'' &= - f_1 \mathring{H}_4' + 4 f_1' \mathring{H}_4 + \frac{8}{3} (2 f_3' C_2 - 3 f_3 C_2') \\ \frac{1}{\sigma} C_6'' &= - f_1 C_6' + 6 f_1' C_6 \\ \frac{1}{\sigma} \mathring{H}_{6a}'' &= - f_1 \mathring{H}_{6a}' + 6 f_1' \mathring{H}_{6a} + 3 (2 g_5' C_2 - 5 g_5 C_2') \\ \frac{1}{\sigma} \mathring{H}_{6b}'' &= - f_1 \mathring{H}_{6b}' + 6 f_1' \mathring{H}_{6b} + 3 (4 f_3' C_4 - 3 f_3 C_4') \\ \frac{1}{\sigma} \mathring{K}_6'' &= - f_1 \mathring{K}_6' + 6 f_1' \mathring{K}_6 + 3 (4 f_3' \mathring{H}_4 - 3 f_3 \mathring{H}_4' + 2 h_5' C_2 - 5 h_5 C_2') \end{aligned}\right\} \tag{12}$$

$$\text{Boundary conditions:} \begin{cases} \eta = 0, \ C_n = \dfrac{1}{n+2} \ \text{and other functions} = 0; \\ \eta = \infty, \quad \text{all functions} = 0. \end{cases}$$

The final expression for Nu is obtained as follows:

$$Nu = -\frac{D}{t_m}\sqrt{\frac{u_1}{\nu}}\left\{t_0\left[F_0' + \frac{4\,u_3}{u_1}F_2'\,x^2 + \cdots\right] + t_2\left[4\,C_2'\,x^2 + 6\,\frac{u_3}{u_1}\,\mathring{H}_4'\,x^4 + \right.\right.$$

$$\left. + 8\left(\frac{u_5}{u_1}\,\mathring{H}_{6a}' + \frac{u_3^2}{u_1^2}\,\mathring{K}_6'\right)x^6 + \cdots\right] + t_4\left[6\,C_4'\,x^4 + 8\,\frac{u_3}{u_1}\,\mathring{H}_{6b}'\,x^6 + \cdots\right] +$$

$$\left. + t_6\left[8\,C_6'\,x^6 + \cdots\right] + \cdots\right\}_{\eta=0}. \tag{13}$$

The set of terms containing t_0 is the solution for the isothermal case, as given in [6] by the author.

B. *Axially symmetrical case*

Owing to the different form of the equation of continuity in this case somewhat different formulas are obtained. The radial distance r is supposed to be of the form

$$r = r_1\,x + r_3\,x^3 + r_5\,x^5 + \cdots \tag{14}$$

and the velocity distribution of the form (2).

Using the equations of ref. [6] we get for a wall temperature distribution of the type (8) in analogy with the case A the functions defined by the following equations (now $\eta = y\sqrt{2\,u_1/\nu}$):

$$\left.\begin{aligned}
&\frac{1}{\sigma}\,C_2'' = -f_1\,C_2' + f_1'\,C_2\\
&\left.\begin{aligned}
&\frac{1}{\sigma}\,C_4'' = -f_1\,C_4' + 2\,f_1'\,C_4\\
&\frac{1}{\sigma}\,\mathring{K}_4'' = -f_1\,\mathring{K}_4' + 2\,f_1'\,\mathring{K}_4 + \frac{4}{3}\,(g_3'\,C_2 - 2\,g_3\,C_2')\\
&\frac{1}{\sigma}\,\mathring{J}_4'' = -f_1\,\mathring{J}_4' + 2\,f_1'\,\mathring{J}_4 + \frac{2}{3}\,(2\,h_3'\,C_2 - 4\,h_3\,C_2' - f_1\,C_2')
\end{aligned}\right]\\
&\left.\begin{aligned}
&\frac{1}{\sigma}\,C_6'' = -f_1\,C_6' + 3\,f_1'\,C_6\\
&\cdots\cdots\cdots\cdots
\end{aligned}\right)
\end{aligned}\right\} \tag{15}$$

Boundary conditions: $\begin{cases}\eta = 0, \quad C_n = \dfrac{2}{n+2} \text{ and other functions} = 0.\\[4pt] \eta = \infty, \text{ all functions} = 0.\end{cases}$

Final expression for Nu:

$$
\begin{aligned}
Nu = -\frac{D}{t_m}\sqrt{\frac{2u_1}{\nu}}\Biggl\{ & t_0\left[F_0' + \frac{2u_3}{u_1}\left(G_2' + \frac{r_3u_1}{r_1u_3}H_2'\right)x^2 + \right. \\
& \left. + \frac{3u_5}{u_1}\left(G_4' + \frac{r_5u_1}{r_1u_5}H_4' + \frac{u_3^2}{u_1u_5}K_4' + \frac{r_3u_3}{r_1u_5}J_4' + \frac{r_3^2u_1}{r_1^2u_5}Q_4'\right)x^4 + \cdots\right] + \\
& + t_2\left[2C_2'x^2 + 3\left(\frac{u_3}{u_1}\mathring{K}_4' + \frac{r_3}{r_1}\mathring{J}_4'\right)x^4 + \cdots\right] + t_4\,[3C_4'x^4 + \cdots] + \\
& + t_6\,[4C_6'x^6 + \cdots] + \cdots\Biggr\}_{\eta=0}. \qquad (16)
\end{aligned}
$$

3. Numerical calculations

In the two-dimensional case numerical calculations were made for the seven universal functions of (12) with a Prandtl number $\sigma = 0.7$ (air).

In the axially symmetrical case the corresponding calculations were made for the first four and five functions of (15) with the Prandtl numbers $\sigma = 0.7$ and $1/0.395$ respectively (the latter is equal to the Schmidt number for naphthalene and is thus applicable to the evaporation of this substance).

For the velocity functions the same values as in [6] were used. Some groups of the equations had common homogeneous parts which were equal to those of certain groups of the equations in [6] and [10]. This made simplifications possible. For such a homogeneous equation one particular solution was computed by the Runge-Kutta method. A second particular solution, linearly independent of the former, was obtained with the aid of the Wronski determinant, which gave a first order differential equation for this second solution. This was possible since the Wronski determinant can be deduced from the next to the highest term [11]. The final solution of the homogeneous equation was obtained by linear combination of those two particular ones. A particular solution of an inhomogeneous equation in the group could be computed by variation of the constants, which could be done by numerical quadrature. Thus an equation of the type

$$z'' + \sigma f_1 z' - m\sigma f_1' z = f \qquad (17)$$

(f_1 and f functions of η) with a particular solution z_1 for the homogeneous part and with the actual boundary conditions has the solution

$$z = z_1\int_0^\eta \frac{F}{z_1^2}\left(\int_0^\zeta \frac{f z_1}{F}\,d\xi\right)d\zeta - z_1\,\frac{\displaystyle\int_0^\infty \frac{F}{z_1^2}\left(\int_0^\zeta \frac{f z_1}{F}\,d\xi\right)d\zeta}{\displaystyle\int_0^\infty \frac{F}{z_1^2}\,d\zeta}\cdot\int_0^\eta \frac{F}{z_1^2}\,d\zeta; \qquad (18)$$

where $$F = e^{-\sigma \int_0^\eta f_1 \, d\eta} \tag{19}$$

For the particular solution z_1 it was convenient to use the initial conditions $\eta = 0$; z_1 = arbitrary value, $z_1' = 0$.

The results obtained for the functions are shown in Tables 1 and 2 respectively and in Figs. 1 and 2. The final decimal place is correct for C_n and has an error of at most a few units for the other functions.

The following values of the derivatives were obtained at the wall ($\eta = 0$) with all decimal places correct:

Two-dimensional:

$$\sigma = 0.7;\ C_2' = -0.2130;\quad C_4' = -0.1757;\quad \mathring{H}_4' = -0.1113;\quad C_6' = -0.1505;$$
$$\mathring{H}_{6a}' = -0.1096;\ \mathring{H}_{6b}' = -0.1006;\quad \mathring{K}_6' = +0.0414.$$

Axially symmetric:

$$\sigma = 0.7;\qquad C_2' = -0.3333;\ C_4' = -0.2657;\ \mathring{J}_4' = -0.0323;\quad \mathring{K}_4' = -0.1653;$$
$$\sigma = 1/0.395;\ C_2' = -0.5306;\ C_4' = -0.4203;\ \mathring{J}_4' = -0.0497;\quad \mathring{K}_4' = -0.2737.$$

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Table 1. Thermal universal functions for two-dimensional symmetrical case. $\sigma = 0.7$.

η	C_2	C_4	$\mathring{H}_4$	C_6	$\mathring{H}_{6a}$	$\mathring{H}_{6b}$	$\mathring{K}_6$
0	0.2500	0.1667	0	0.1250	0	0	0
0.2	0.2079	0.1322	−0.0214	0.0956	−0.0211	−0.0190	0.0084
0.4	0.1685	0.1010	−0.0391	0.0698	−0.0385	−0.0329	0.0171
0.6	0.1334	0.0748	−0.0511	0.0490	−0.0502	−0.0402	0.0256
0.8	0.1033	0.0538	−0.0574	0.0333	−0.0562	−0.0417	0.0331
1.0	0.0784	0.0378	−0.0586	0.0220	−0.0571	−0.0390	0.0390
1.2	0.0583	0.0260	−0.0559	0.0142	−0.0542	−0.0340	0.0428
1.4	0.0425	0.0174	−0.0506	0.0089	−0.0487	−0.0281	0.0444
1.6	0.0304	0.0115	−0.0438	0.0055	−0.0420	−0.0221	0.0437
1.8	0.0213	0.0074	−0.0365	0.0033	−0.0347	−0.0169	0.0415
2.0	0.0147	0.0047	−0.0294	0.0020	−0.0279	−0.0123	0.0376
2.2	0.0099	0.0029	−0.0230	0.0012	−0.0215	−0.0089	0.033
2.4	0.0065	0.0018	−0.0174	0.0006	−0.0163	−0.0064	0.028
2.6	0.0042	0.0011	−0.0128	0.0004	−0.0118	−0.0043	0.023
2.8	0.0027	0.0006	−0.0091	0.0002	−0.0085	−0.0027	0.018
3.0	0.0016	0.0004	−0.0064	0.0001	−0.0061	−0.0017	0.016
3.2	0.0010	0.0002	−0.0043	0.0000	−0.0041	−0.0011	0.011
3.4	0.0006	0.0001	−0.0028		−0.0026	−0.0008	0.008
3.6	0.0003	0.0000	−0.0018		−0.0016	−0.0005	0.005
3.8	0.0002		−0.0011		−0.0009	−0.0003	0.004
4.0	0.0001		−0.0007		−0.0004	−0.0001	0.002
4.2	0.0000		−0.0004		−0.0002	−0.0000	0.001
4.4			−0.0002		−0.0000		0.001
4.6			−0.0001				0.000
4.8			−0.0001				
5.0			−0.0000				

Table 2. Thermal universal functions for axially symmetrical case. $\sigma = 0.7$.

η	C_2	C_4	$\mathring{J}_4$	$\mathring{K}_4$
0	0.5000	0.3333	0	0
0.2	0.4338	0.2807	−0.0064	−0.0325
0.4	0.3698	0.2309	−0.0126	−0.0617
0.6	0.3098	0.1857	−0.0183	−0.0855
0.8	0.2550	0.1463	−0.0230	−0.1027
1.0	0.2062	0.1129	−0.0266	−0.1128
1.2	0.1637	0.0854	−0.0288	−0.1162
1.4	0.1275	0.0633	−0.0296	−0.1137
1.6	0.0975	0.0460	−0.0290	−0.1065
1.8	0.0731	0.0328	−0.0272	−0.0960
2.0	0.0537	0.0229	−0.0246	−0.0836
2.2	0.0387	0.0157	−0.0215	−0.0703
2.4	0.0273	0.0105	−0.0181	−0.0573
2.6	0.0188	0.0069	−0.0147	−0.0453
2.8	0.0127	0.0045	−0.0116	−0.0348
3.0	0.0084	0.0028	−0.0089	−0.0259
3.2	0.0054	0.0017	−0.0066	−0.0188
3.4	0.0034	0.0010	−0.0047	−0.0132
3.6	0.0021	0.0006	−0.0033	−0.0090
3.8	0.0013	0.0003	−0.0022	−0.0060
4.0	0.0007	0.0002	−0.0014	−0.0038
4.2	0.0004	0.0001	−0.0009	−0.0024
4.4	0.0002	0.0000	−0.0006	−0.0014
4.6	0.0001		−0.0003	−0.0008
4.8	0.0000		−0.0002	−0.0005
5.0			−0.0001	−0.0002
5.2			−0.0000	−0.0001
5.4				−0.0000

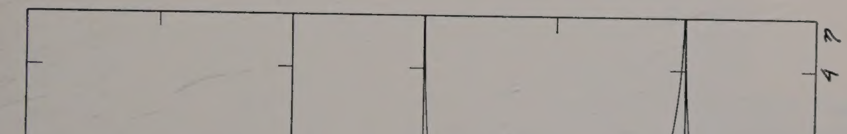

Fig. 1. Thermal universal functions for two-dimensional symmetrical case. $\sigma = 0.7$.

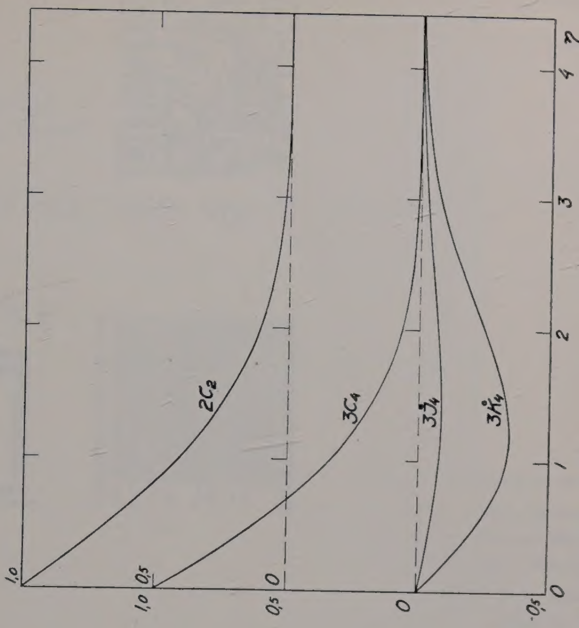

Fig. 2. Thermal universal functions for axially symmetrical case. $\sigma = 0.7$.

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